The completion of L-topological groups

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Abstract

The target in this paper, is to extend an L-topological group to a complete L-topological group, and so giving the notion of the completion of an L-topological group. In the way, we have introduced the notion of the completion of an L-uniform space.

Keywords: L-topological groups; complete L-topological groups; L-uniform spaces; complete L-uniform spaces; $\mathcal{U}$-cauchy filters; L-filters; L-topological spaces.

1. Introduction

In this paper, we gave new notions of L-filter, L-uniform space and L-topological group. We defined, in an L-uniform space $(X,\mathcal{U})$, a $\mathcal{U}$-cauchy filter and have shown when $(X,\mathcal{U})$ to be a complete L-uniform space, and also how an L-topological group (G,τ) to be complete. Finally, the completion of an L-uniform space and the completion of an L-topological group are investigated.

In Section 2, we recall some results of L-filters and L-neighborhood filters defined by Gähler in [11, 13, 14]. Also, we have defined the product of two L-sets and the product of two L-filters.

In Section 3, we have defined in an L-uniform space $(X,\mathcal{U})$, a new notion of L-filter called *$\mathcal{U}$-cauchy filter*. We showed that any convergent L-filter is a $\mathcal{U}$-cauchy filter and the converse holds in the complete L-uniform spaces.

Section 4 is devoted to show how to extend an L-uniform space to a complete L-uniform space, and so the completion of an L-uniform space here is given as a reduced extension L-uniform space with a complete L-uniform structure.

In Section 5, using the L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$ defined on the L-topological group (G,τ) which are compatible with τ as in [8], we shall define the notion of *complete* L-topological group. A complete separated L-topological group (H,σ) in which (G,τ) is a dense subgroup will be called a completion of (G,τ).

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2. On L-filters

In this section, we recall and show some results concerning L-filters needed in the paper. Denote by L^X the set of all L-subsets of a non-empty set X, where L is a complete chain with different least and greatest elements 0 and 1, respectively [19]. For each L-set $\lambda \in L^X$, let λ' denote the complement of λ, defined by $\lambda'(x) = \lambda(x)'$ for all $x \in X$. For all $x \in X$ and $\alpha \in L_0$, the L-subset x_α of X whose value α at x and 0 otherwise is called an *L-point* in X and the constant L-subset of X with value α will be denoted by $\overline{\alpha}$.

***L*-filters.** By an *L-filter* on a non-empty set X we mean [13] a mapping $\mathcal{M} : L^X \to L$ such that $\mathcal{M}(\overline{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$, and also $\mathcal{M}(\lambda \wedge \mu) = \mathcal{M}(\lambda) \wedge \mathcal{M}(\mu)$ for all $\lambda, \mu \in L^X$. $\mathcal{M}$ ia called *homogeneous* [11] if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. If $\mathcal{M}$ and $\mathcal{N}$ are L-filters on X, $\mathcal{M}$ is called *finer* than $\mathcal{N}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\lambda) \geq \mathcal{N}(\lambda)$ holds for all $\lambda \in L^X$.

Let $\mathcal{F}_L X$ denote the set of all L-filters on X, $f : X \to Y$ a mapping and $\mathcal{M}$, $\mathcal{N}$ are L-filters on X, Y, respectively. Then the *image* of $\mathcal{M}$ and the *preimage* of $\mathcal{N}$ with respect to f are the L-filters $\mathcal{F}_L f(\mathcal{M})$ on Y and $\mathcal{F}_L^- f(\mathcal{N})$ on X defined by $\mathcal{F}_L f(\mathcal{M})(\mu) = \mathcal{M}(\mu \circ f)$ for all $\mu \in L^Y$ and $\mathcal{F}_L^- f(\mathcal{N})(\lambda) = \bigvee_{\mu \circ f \leq \lambda} \mathcal{N}(\mu)$ for all $\lambda \in L^X$, respectively. For each mapping $f : X \to Y$ and each L-filter $\mathcal{N}$ on Y, for which the preimage $\mathcal{F}_L^- f(\mathcal{N})$ exists, we have $\mathcal{F}_L f(\mathcal{F}_L^- f(\mathcal{N})) \leq \mathcal{N}$. Moreover, for each L-filter $\mathcal{M}$ on X, the inequality $\mathcal{M} \leq \mathcal{F}_L^- f(\mathcal{F}_L f(\mathcal{M}))$ holds [13].

For each non-empty set A of L-filters on X, the supremum $\bigvee_{\mathcal{M} \in A} \mathcal{M}$ with respect to the finer relation of L-filters exists and we have

$$(\bigvee_{\mathcal{M} \in A} \mathcal{M})(f) = \bigwedge_{\mathcal{M} \in A} \mathcal{M}(f)$$

for all $f \in L^X$. The infimum $\bigwedge_{\mathcal{M} \in A} \mathcal{M}$ of A exists *if and only if* for each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of A we have $\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n) \leq \sup(\lambda_1 \wedge \cdots \wedge \lambda_n)$ for all $\lambda_1, \ldots, \lambda_n \in L^X$ [11]. If the infimum of A exists, then for each $\lambda \in L^X$ and n as a positive integer we have

$$(\bigwedge_{\mathcal{M} \in A} \mathcal{M})(\lambda) = \bigvee_{\substack{\lambda_1 \wedge \cdots \wedge \lambda_n \leq \lambda, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in A}} (\mathcal{M}_1(\lambda_1) \wedge \cdots \wedge \mathcal{M}_n(\lambda_n)).$$

By a *filter* on X we mean a non-empty subset $\mathcal{F}$ of L^X which does not contain $\overline{0}$ and closed under finite infima and super sets [17]. For each L-filter $\mathcal{M}$ on X, the subset $\alpha\text{-pr}\,\mathcal{M}$ of L^X defined by: $\alpha\text{-pr}\,\mathcal{M} = \{\lambda \in L^X \mid \mathcal{M}(\lambda) \geq \alpha\}$ is a filter on X.

A family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X is called *valued L-filter base* on X [13] if the following conditions are fulfilled:

(V1) $\lambda \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup \lambda$.

(V2) For all $\alpha, \beta \in L_0$ and all L-sets $\lambda \in \mathcal{B}_\alpha$ and $\mu \in \mathcal{B}_\beta$, if even $\alpha \wedge \beta > 0$ holds, then there are a $\gamma \geq \alpha \wedge \beta$ and an L-set $\nu \leq \lambda \wedge \mu$ such that $\nu \in \mathcal{B}_\gamma$.

Each valued L-filter base $(\mathcal{B}_\alpha)_{\alpha\in L_0}$ on a set X defines an L-filter $\mathcal{M}$ on X by: $\mathcal{M}(\lambda) = \bigvee_{\mu\in\mathcal{B}_\alpha,\,\mu\leq\lambda} \alpha$ for all $\lambda \in L^X$. On the other hand, each L-filter $\mathcal{M}$ can be generated by many valued L-filter bases, and among them the greatest one $(\alpha\text{-pr}\,\mathcal{M})_{\alpha\in L_0}$.

Proposition 2.1 [13] *There is a one-to-one correspondence between the L- filters $\mathcal{M}$ on X and the families $(\mathcal{M}_\alpha)_{\alpha\in L_0}$ of prefilters on X which fulfill the following conditions:*

(1) $f \in \mathcal{M}_\alpha$ *implies* $\alpha \leq \sup f$.

(2) $0 < \alpha \leq \beta$ *implies* $\mathcal{M}_\alpha \supseteq \mathcal{M}_\beta$.

(3) *For each* $\alpha \in L_0$ *with* $\bigvee_{0<\beta<\alpha} \beta = \alpha$ *we have* $\bigcap_{0<\beta<\alpha} \mathcal{M}_\beta = \mathcal{M}_\alpha$.

This correspondence is given by $\mathcal{M}_\alpha = \alpha\text{-pr}\,\mathcal{M}$ *for all* $\alpha \in L_0$ *and* $\mathcal{M}(f) = \bigvee_{g\in\mathcal{M}_\alpha,\,g\leq f} \alpha$ *for all* $f \in L^X$.

***L*-neighborhood filters.** In the following, the topology in sense of [10, 16] will be used which will be called L-topology. int_τ and cl_τ denote the interior and the closure operators with respect to the L-topology τ, respectively. For each L-topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by: $\mathcal{N}(x)(\lambda) = \text{int}_\tau\lambda(x)$ for all $\lambda \in L^X$ is an L-filter on X, called the *L-neighborhood filter* of the space (X, τ) at x, and for short is called a τ-neighborhood filter at x. The mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\lambda) = \lambda(x)$ for all $\lambda \in L^X$ is a homogeneous L-filter on X. Let (X, τ) and (Y, σ) be two L-topological spaces. Then the mapping $f : (X, \tau) \to (Y, \sigma)$ is called *L-continuous* (or *(τ, σ)-continuous*) provided $\text{int}_\sigma\mu \circ f \leq \text{int}_\tau(\mu \circ f)$ for all $\mu \in L^Y$. An L-filter $\mathcal{M}$ is said to *converge to* $x \in X$, denoted by $\mathcal{M} \underset{\tau}{\to} x$, if $\mathcal{M} \leq \mathcal{N}(x)$ [14]. The L-neighborhood filter $\mathcal{N}(F)$ at an ordinary subset F of X is the L-filter on X defined, by the authors in [3], by means of $\mathcal{N}(x)$, $x \in F$ as:

$$\mathcal{N}(F) = \bigvee_{x\in F} \mathcal{N}(x).$$

The L-filter $\dot{F}$ is defined by $\dot{F} = \bigvee_{x\in F} \dot{x}$. $\dot{F} \leq \mathcal{N}(F)$ holds for all $F \subseteq X$.

Lemma 2.1 [14] *Let (X, τ) and (Y, σ) be two L-topological spaces and $\mathcal{M}$ an L-filter on X, and let $f : X \to Y$ be a (τ, σ)-continuous mapping. Then $\mathcal{M} \underset{\tau}{\to} x$ implies that $\mathcal{F}_L f(\mathcal{M}) \underset{\sigma}{\to} f(x)$.*

Firstly, let us give this important definition.

For $\lambda, \mu \in L^X$, let $\lambda \times \mu : X \times X \to L$ be the L-set defined as follows:

$$(\lambda \times \mu)(x, y) = \lambda(x) \wedge \mu(y) \tag{2.1}$$

for all $x, y \in X$.

Remark 2.1 For all $\lambda, \mu, \xi, \eta \in L^X$, we have

$$(\lambda \wedge \mu) \times (\xi \wedge \eta) \ = \ (\lambda \times \xi) \wedge (\mu \times \eta) \ = \ (\lambda \times \eta) \wedge (\mu \times \xi).$$

Proposition 2.2 *For any two L-filters $\mathcal{L}, \mathcal{M}$ on X, the mapping $\mathcal{L} \times \mathcal{M} : L^{X \times X} \to L$ defined by*

$$(\mathcal{L} \times \mathcal{M})(u) \ = \ \bigvee_{\lambda \times \mu \leq u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \tag{2.2}$$

for all $u \in L^{X \times X}$ is an L-filter on $X \times X$.

Proof. From (2.1) and that $\mathcal{L}, \mathcal{M}$ are L-filters, we get that

$$(\mathcal{L} \times \mathcal{M})(\widetilde{\alpha}) \ = \ \bigvee_{\lambda \times \mu \leq \widetilde{\alpha}} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \leq \alpha.$$

Moreover, $(\mathcal{L} \times \mathcal{M})(\widetilde{1}) = 1$.

From Remark 2.1 and for all $u, v \in L^{X \times X}$, we get that

$$\begin{aligned}
(\mathcal{L} \times \mathcal{M})(u) \wedge (\mathcal{L} \times \mathcal{M})(v) \ &= \ \bigvee_{\lambda \times \mu \leq u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \wedge \bigvee_{\xi \times \eta \leq v} (\mathcal{L}(\xi) \wedge \mathcal{M}(\eta)) \\
&= \ \bigvee_{\lambda \times \mu \leq u, \, \xi \times \eta \leq v} (\mathcal{L}(\lambda \wedge \xi) \wedge \mathcal{M}(\mu \wedge \eta)) \\
&\leq \ \bigvee_{(\lambda \wedge \xi) \times (\mu \wedge \eta) \leq u \wedge v} (\mathcal{L}(\lambda \wedge \xi) \wedge \mathcal{M}(\mu \wedge \eta)) \\
&= \ (\mathcal{L} \times \mathcal{M})(u \wedge v).
\end{aligned}$$

Also,

$$\begin{aligned}
(\mathcal{L} \times \mathcal{M})(u \wedge v) \ &= \ \bigvee_{\lambda \times \mu \leq u \wedge v} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu) \\
&\leq \ \bigvee_{\lambda \times \mu \leq u, \, \lambda \times \mu \leq v} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \\
&= \ \bigvee_{\lambda \times \mu \leq u} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \ \wedge \ \bigvee_{\lambda \times \mu \leq v} (\mathcal{L}(\lambda) \wedge \mathcal{M}(\mu)) \\
&= \ (\mathcal{L} \times \mathcal{M})(u) \wedge (\mathcal{L} \times \mathcal{M})(v).
\end{aligned}$$

Hence, $(\mathcal{L} \times \mathcal{M})$ is an L-filter on $X \times X$. $\Box$

Here, we prove the following result.

Lemma 2.2 *Let $\mathcal{L}$ and $\mathcal{M}$ be L-filters on X, and let $(\mathcal{L}_\alpha)_{\alpha \in L_0}$ and $(\mathcal{M}_\alpha)_{\alpha \in L_0}$ be the families of prefilters on X correspond, according to* Proposition 2.1, *$\mathcal{L}$ and $\mathcal{M}$, respectively. Then the family $(\mathcal{K}_\alpha)_{\alpha \in L_0}$ of subsets $\mathcal{K}_\alpha$ of $L^{X \times X}$, where*

$$\mathcal{K}_\alpha \ = \ \{\lambda \times \mu \mid \lambda \in \mathcal{L}_\alpha, \, \mu \in \mathcal{M}_\alpha\}, \tag{2.3}$$

is a family of prefilters on $X \times X$ corresponds the L-filter $\mathcal{L} \times \mathcal{M}$.

Proof. Firstly, we show that, for all $\alpha \in L_0$, $\mathcal{K}_\alpha$ is a prefilter on $X \times X$. For any $\alpha \in L_0$, we have $\mathcal{K}_\alpha = \{\lambda \times \mu \mid \lambda \in \mathcal{L}_\alpha,\, \mu \in \mathcal{M}_\alpha\}$ is non-empty, where $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$ are non-empty for all $\alpha \in L_0$. Also, $\overline{0}$ does not exist in $\mathcal{L}_\alpha$ or $\mathcal{M}_\alpha$ implies that $\overline{0} \notin \mathcal{K}_\alpha$ for all $\alpha \in L_0$. From Remark 2.1 and from that $\mathcal{L}_\alpha$ and $\mathcal{M}_\alpha$ are prefilters, we get for all $u, v \in \mathcal{K}_\alpha$ and $w \geq v$ that $u \wedge v \in \mathcal{K}_\alpha$ and $w \in \mathcal{K}_\alpha$ for all $\alpha \in L_0$. That is, $\mathcal{K}_\alpha$, for all $\alpha \in L_0$, is a prefilter on $X \times X$.

Let $u \in \mathcal{K}_\alpha$. Then $u = \lambda \times \mu$, where $\lambda \in \mathcal{L}_\alpha$ and $\mu \in \mathcal{M}_\alpha$, which implies that $\alpha \leq \sup\lambda$, $\alpha \leq \sup\mu$, and $\alpha \leq \sup(\lambda \times \mu) = \sup u$, that is, condition (1) of Proposition 2.1 holds.

Let $0 < \alpha \leq \beta$ and $u \in \mathcal{K}_\beta$. Then $u = \lambda \times \mu$, where $\lambda \in \mathcal{L}_\beta$ and $\mu \in \mathcal{M}_\beta$, which implies, from $\mathcal{L}_\alpha \supseteq \mathcal{L}_\beta$ and $\mathcal{M}_\alpha \supseteq \mathcal{M}_\beta$, that $\lambda \in \mathcal{L}_\alpha$ and $\mu \in \mathcal{M}_\alpha$, that is, $u \in \mathcal{K}_\alpha$ and condition (2) of Proposition 2.1 is fulfilled.

Since $\bigcap\limits_{0<\beta<\alpha} \mathcal{L}_\beta = \mathcal{L}_\alpha$ and $\bigcap\limits_{0<\beta<\alpha} \mathcal{M}_\beta = \mathcal{M}_\alpha$, we get that

$$\begin{aligned}
\bigcap_{0<\beta<\alpha} \mathcal{K}_\beta &= \bigcap_{0<\beta<\alpha} \{\lambda \times \mu \mid \lambda \in \mathcal{L}_\beta, \mu \in \mathcal{M}_\beta\} \\
&= \{\lambda \times \mu \mid \lambda \in \bigcap_{0<\beta<\alpha} \mathcal{L}_\beta, \mu \in \bigcap_{0<\beta<\alpha} \mathcal{M}_\beta\} \\
&= \{\lambda \times \mu \mid \lambda \in \mathcal{L}_\alpha, \mu \in \mathcal{M}_\alpha\} \\
&= \mathcal{K}_\alpha,
\end{aligned}$$

which means that condition (3) of Proposition 2.1 holds.

Hence, there is a one - to - one correspondence between the family $(\mathcal{K}_\alpha)_{\alpha \in L_0}$ of the prefilters on $X \times X$, defined by (2.3), and the L-filter $\mathcal{L} \times \mathcal{M}$ on $X \times X$, according to Proposition 2.1, where

$$(\mathcal{L} \times \mathcal{M})(u) = \bigvee_{v \in \mathcal{K}_\alpha,\, v \leq u} \alpha \quad \text{and} \quad \alpha\text{-pr}\,(\mathcal{L} \times \mathcal{M}) = \mathcal{K}_\alpha$$

for all $u \in L^{X \times X}$ and for all $\alpha \in L_0$. □

3. $\mathcal{U}$-cauchy filters

This section is devoted to speak of the cauchy filters in the L-uniform spaces defined in [15].

***L*-uniform spaces.** An L-filter $\mathcal{U}$ on $X \times X$ is called L-uniform structure on X [15] if the following conditions are fulfilled:

(U1) $(x,x)^{\bullet} \leq \mathcal{U}$ for all $x \in X$;

(U2) $\mathcal{U} = \mathcal{U}^{-1}$;

(U3) $\mathcal{U} \circ \mathcal{U} \leq \mathcal{U}$.

Where $(x,x)^{\bullet}(u) = u(x,x)$, $\mathcal{U}^{-1}(u) = \mathcal{U}(u^{-1})$ and $(\mathcal{U} \circ \mathcal{U})(u) = \bigvee\limits_{v \circ w \leq u} \mathcal{U}(v \wedge w)$ for all $u \in L^{X \times X}$, and $u^{-1}(x,y) = u(y,x)$ and $(v \circ w)(x,y) = \bigvee\limits_{z \in X} (\, w(x,z) \wedge v(z,y))$ for all $x, y \in X$.

A set X equipped with an L-uniform structure $\mathcal{U}$ is called an *L-uniform space.*

To each L-uniform structure $\mathcal{U}$ on X is associated a stratified L-topology $\tau_{\mathcal{U}}$. The related interior operator $\text{int}_{\mathcal{U}}$ is given by:

$$(\text{int}_{\mathcal{U}}\lambda)(x) \;=\; \mathcal{U}[\dot{x}](\lambda)$$

for all $x \in X$ and all $\lambda \in L^X$, where $\mathcal{U}[\dot{x}](\lambda) = \bigvee_{u[\mu]\leq\lambda} (\mathcal{U}(u) \wedge \mu(x))$ and $u[\mu](x) = \bigvee_{y\in X} (\mu(y) \wedge u(y,x))$. For all $x \in X$ we have

$$\mathcal{U}[\dot{x}] \;=\; \mathcal{N}(x)$$

where $\mathcal{N}(x)$ is the L-neighborhood filter of the space $(X, \tau_{\mathcal{U}})$ at x. That is, an L-filter $\mathcal{M}$ in an L-uniform space $(X,\mathcal{U})$ is said to converge to $x \in X$ if $\mathcal{M} \leq \mathcal{U}[\dot{x}]$.

Let $\mathcal{U}$ be an L-uniform structure on a set X. Then $u \in L^{X\times X}$ is called a *surrounding* provided $\mathcal{U}(u) \geq \alpha$ for some $\alpha \in L_0$ and $u = u^{-1}$ [8].

A subset $A \subseteq X$, for a surrounding u in $(X,\mathcal{U})$, is called *small of order* u if $u(x,y) \geq \alpha$ for all $x, y \in A$ and for some $\alpha \in L_0$.

Definition 3.1 In an L-uniform space $(X,\mathcal{U})$, an L-filter $\mathcal{M}$ on X is said to be a *$\mathcal{U}$-cauchy filter* provided for any surrounding u, there exists a set $B \subseteq X$ such that $\mathcal{M} \leq \dot{B}$ and B is small of order u.

Now, we have the following expected result for the convergent L-filters.

Proposition 3.1 *Every convergent L-filter in an L-uniform space $(X,\mathcal{U})$ is a $\mathcal{U}$-cauchy filter.*

Proof. Let $\mathcal{M}$ be an L-filter on X which converges to $x \in X$. Since $\mathcal{M} \leq \mathcal{U}[\dot{x}]$, then we can choose a set $B \subseteq X$ such that $\mathcal{M} \leq \dot{B} = \mathcal{U}[\dot{x}]$, that is,

$$\mathcal{M}(\lambda) \geq \bigvee_{u[\mu]\leq\lambda} (\mathcal{U}(u) \wedge \mu(x)) = \bigwedge_{y\in B} \lambda(y) = \dot{B}(\lambda)$$

for all $\lambda \in L^X$. Since $(x,x)^{\bullet} \leq \mathcal{U}$ for all $x \in X$, then $u(x,x) \geq \mathcal{U}(u) \geq \alpha$ for any surrounding u and for some $\alpha \in L_0$, that is, $u(x,x) \geq \alpha$ for all $x \in X$ and for some $\alpha \in L_0$. Now, $x \in B$ where $\dot{x} \leq \mathcal{U}[\dot{x}] = \dot{B}$. Also, for any $y \in B$ we get that $\bigvee_{u[\mu]\leq\lambda} (\alpha \wedge \mu(x)) \leq \lambda(y)$, for which $\bigvee_{z}(u(z,y) \wedge \mu(z)) \leq \lambda(y)$, and so $\alpha \wedge \mu(x) \leq u(x,y) \wedge \mu(x) \leq \lambda(y)$, and thus for all $x, y \in B$, we have $u(x,y) \geq \alpha$ for some $\alpha \in L_0$ and $\mathcal{M} \leq \dot{B}$. Hence, there is a set $B \subseteq X$ small of order any surrounding u in $(X,\mathcal{U})$ and $\mathcal{M} \leq \dot{B}$, and therefore $\mathcal{M}$ is a $\mathcal{U}$-cauchy filter on X. □

Let A be a subset of a set X, $\mathcal{U}$ an L-uniform structure on X and $i : A \hookrightarrow X$ the inclusion mapping of A into X. Then the initial L-uniform structure $\mathcal{F}_L^{-}(i \times i)(\mathcal{U})$ of $\mathcal{U}$ with respect to i, denoted by $\mathcal{U}_A$, is called an *L-uniform substructure* of $\mathcal{U}$ and $(A,\mathcal{U}_A)$ an *L-uniform subspace* of $(X,\mathcal{U})$ [4].

In particular, we have the following result.

Lemma 3.1 *Let $(X,\mathcal{U})$ be an L-uniform space and A a non-empty subset of X. Then an L-filter on A is a $\mathcal{U}_A$-cauchy filter* if and only if *it is a $\mathcal{U}$-cauchy filter.*

Proof. Let $\mathcal{M}$ be a $\mathcal{U}_A$-cauchy filter on A, then there exists $B \subseteq A$ with $\mathcal{M} \leq \dot{B}$ and B is small of order any surrounding u_A in $(A,\mathcal{U}_A)$, which means that there is $B \subseteq A \subseteq X$ such that $\mathcal{M} \leq \dot{B}$ and $u_A(x,y) \geq \alpha$ for all $x, y \in B$ and for some $\alpha \in L_0$, that is, for any surrounding u in $(X,\mathcal{U})$,

$$u(x,y) = (u \circ (i \times i))(x,y) = u_A(x,y) \geq \alpha$$

for all $x, y \in B$ and for some $\alpha \in L_0$, and then $\mathcal{M} \leq \dot{B}$ and $B \subseteq X$ is small of order any surrounding u in $(X,\mathcal{U})$. Hence, $\mathcal{M}$ is a $\mathcal{U}$-cauchy filter.

Conversely; there exists $B \subseteq A \subseteq X$ with $\mathcal{M} \leq \dot{B}$ and B is small of order any surrounding u in $(X,\mathcal{U})$, that is, $u(x,y) \geq \alpha$ for all $x, y \in B$ and for some $\alpha \in L_0$, which means that, for every surrounding u_A in $(A,\mathcal{U}_A)$,

$$u_A(x,y) = (u \circ (i \times i))(x,y) = u(x,y) \geq \alpha$$

for all $x, y \in B$ and for some $\alpha \in L_0$. Hence, $\mathcal{M} \leq \dot{B}$ and $B \subseteq A$ is small of order any surrounding u_A in $(A,\mathcal{U}_A)$, and thus $\mathcal{M}$ is a $\mathcal{U}_A$-cauchy filter. $\square$

A mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ between L-uniform spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ is said to be *L-uniformly continuous* (or *$(\mathcal{U},\mathcal{V})$-continuous*) provided

$$\mathcal{F}_L(f \times f)(\mathcal{U}) \;\leq\; \mathcal{V}$$

holds.

We shall use this result.

Lemma 3.2 *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be L-uniform spaces and $f : X \to Y$ a $(\mathcal{U},\mathcal{V})$-continuous mapping. If $\mathcal{M}$ is a $\mathcal{U}$-cauchy filter, then $\mathcal{F}_L f(\mathcal{M})$ is a $\mathcal{V}$-cauchy filter.*

Proof. $\mathcal{M}$ is a $\mathcal{U}$-cauchy filter on X means that there exists $B \subseteq X$ such that $\mathcal{M} \leq \dot{B}$ and B is small of order any surrounding u in $(X,\mathcal{U})$, that is, $\mathcal{M} \leq \dot{B}$ and $u(x,y) \geq \alpha$ for all $x, y \in B$ and for some $\alpha \in L_0$, which implies that,

$$\mathcal{F}_L f(\mathcal{M}) \leq \mathcal{F}_L f(\dot{B}) = (f(\dot{B}))$$

for the set $f(B) \subseteq Y$. Let v be a surrounding in $(Y,\mathcal{V})$, then from being f is $(\mathcal{U},\mathcal{V})$-continuous, we have

$$\alpha \leq \mathcal{V}(v) \leq \mathcal{U}(v \circ (f \times f)) = \mathcal{F}_L(f \times f)(\mathcal{U})(v)$$

for some $\alpha \in L_0$, and $v = v^{-1}$ implies that $(v \circ (f \times f))^{-1} = v^{-1} \circ (f \times f) = v \circ (f \times f)$, that is, $u = v \circ (f \times f)$ is a surrounding in $(X,\mathcal{U})$, which means that

$$\alpha \leq u(x,y) = (v \circ (f \times f))(x,y) = v(f(x), f(y))$$

for all $f(x), f(y) \in f(B)$ and for some $\alpha \in L_0$. Hence, $\mathcal{F}_L f(\mathcal{M}) \leq (f(\dot{B}))$ for the set $f(B) \subseteq Y$ and $f(B)$ is small of order every surrounding in $(Y,\mathcal{V})$, and thus $\mathcal{F}_L f(\mathcal{M})$ is a $\mathcal{V}$-cauchy filter. $\square$

4. The completion of L-uniform spaces

Firstly, we give these general notes.

If (Y, σ) is an L-topological space and X is a non-empty subset of Y, then the initial L-topology of σ, with respect to the inclusion mapping $i : X \hookrightarrow Y$, is the L-topology $i^{-1}(\sigma) = \{i^{-1}(\lambda) \mid \lambda \in \sigma\}$ on X and is denoted by σ_X.

An L-topological space (Y, σ) is called an *extension* of the L-topological space (X, τ) if $X \subseteq Y$, $\tau = \sigma_X$ and X is σ-dense in Y.

The extension (Y, σ) of (X, τ) is called *reduced* if for any $x \neq y$ in Y and $x \in Y \backslash X$, we have $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$, where $\mathcal{N}_\sigma(x)$ denotes the L-neighborhood filter of (Y, σ) at a point $x \in Y$.

In [2, 3, 7, 8], we have introduced and studied the notion of GT_i-spaces for all $i = 0, 1, 2, 3, 3\frac{1}{2}, 4$.

GT_i-spaces. An L-topological space (X, τ) is called [2, 3, 7]:

(1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ or $\dot{y} \nleq \mathcal{N}(x)$.

(2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ and $\dot{y} \nleq \mathcal{N}(x)$.

(3) GT_2 if for all $x, y \in X$ with $x \neq y$, we have $\mathcal{M} \nleq \mathcal{N}(x)$ or $\mathcal{M} \nleq \mathcal{N}(y)$ for all L-filters $\mathcal{M}$ on X.

(4) *regular* if for all $x \notin F$ and $F = \mathrm{cl}_\tau F$, we have $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist.

(5) GT_3 if it is GT_1 and regular.

(6) *completely regular* if for all $x \notin F \in \tau'$, there exists a L- continuous mapping $f : (X, \tau) \to (I_L, \Im)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \in F$.

(7) $GT_{3\frac{1}{2}}$ (or *L-Tychonoff*) if it is GT_1 and completely regular.

Denote by *GT_i-space* the L- topological space which is GT_i, $i = 0, 1, 2, 3, 3\frac{1}{2}$.

Proposition 4.1 [2, 3, 7] *Every GT_i-space is GT_{i-1}-space for each $i = 1, 2, 3$, and every $GT_{3\frac{1}{2}}$-space is a GT_3-space.*

Lemma 4.1 *If the extension (Y, σ) of (X, τ) is a GT_0-space, then (Y, σ) is a reduced extension of (X, τ).*

Proof. Clear. □

Lemma 4.2 *For a GT_0-space (X, τ), the reduced extension (Y, σ) also is a GT_0-space.*

Proof. For all $x \neq y$ in $Y \backslash X$, we have $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$. Also for all $x \neq y$ in X, we have $\mathcal{N}_\tau(x) \neq \mathcal{N}_\tau(y)$. Hence, for all $x \neq y$ in Y we get that $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$, and thus (Y, σ) is a GT_0-space. □

Remark 4.1 Let (X, τ) be an L-topological space and $X \subseteq Y$. If we succeed in defining an L-topology σ on Y such that (Y, σ) is an extension of (X, τ), then X is a σ-dense in Y implies that every σ-neighborhood of each $y \in Y$ intersects X, hence the infimum $\mathcal{N}_\sigma(y) \wedge \dot{X}$ exists where, for all $f, g \in L^X$, $\text{int}_\sigma f(y) = f(x)$ for some $x \in X$ implies $\text{int}_\sigma f(y) \wedge \bigwedge\limits_{x\in X} g(x) \leq f(x)$ for some $x \in X$ and also $\text{int}_\sigma f(y) \wedge \bigwedge\limits_{x\in X} g(x) \leq g(x)$ for all $x \in X$, and thus $\text{int}_\sigma f(y) \wedge \bigwedge\limits_{x\in X} g(x) \leq \sup(f \wedge g)$ for all $f, g \in L^X$.

Definition 4.1 Let (X, τ), (Y, σ) be two L- topological spaces and (Y, σ) an extension of (X, τ). Then the L-filter $\mathcal{N}_\sigma(x) \wedge \dot{X}$ on X, denoted by $\mathcal{M}_x$, will be called a *trace filter at* $x \in Y$ into Y and $\mathcal{M}_x = \mathcal{N}_\tau(x)$ whenever $x \in X$. Clearly, $\mathcal{M}_x \underset{\sigma}{\to} x$.

Definition 4.2 Let (X, τ) and (Y, σ) be two L-topological spaces, (X', τ^*) an extension of (X, τ) and let $f : X \to Y$ be a (τ, σ)-continuous mapping. Then the restriction mapping $g|_X$ on X of the (τ^*, σ)-continuous mapping $g : X' \to Y$, which coincides with f, is called a *continuous extension* of f into X'.

Remark 4.2 Let (X, τ) and (Y, σ) be two L-topological spaces, (X', τ^*) an extension of (X, τ), $f : X \to Y$ a mapping and $\mathcal{M}_x = \mathcal{N}_{\tau^*}(x) \wedge \dot{X}$ a trace filter on X at $x \in X'$. For the existence of a continuous extension $g : X' \to Y$, it is necessary that f is (τ, σ)-continuous and $\mathcal{F}_L f(\mathcal{M}_x) \underset{\sigma}{\to} x$ for a trace filter $\mathcal{M}_x$ at $x \in X'$. If (Y, σ) is a regular space, then these conditions also are sufficient. It is clear that $\mathcal{M}_x \underset{\tau^*}{\to} x$.

Lemma 4.3 *With the notations in* Remark 4.2*, let* $g_1 : X' \to Y$ *and* $g_2 : X' \to Y$ *be* (τ^*, σ)*-continuous,* (Y, σ) *is a* GT_2*-space and* $g_1|_X = g_2|_X = f$*. Then* $g_1 = g_2$*.*

Proof. Let $x \in X'$ be arbitrary and $\mathcal{M}_x \underset{\tau^*}{\to} x$. From Lemma 2.1, we get that $\mathcal{F}_L g_1(\mathcal{M}_x) \underset{\sigma}{\to} g_1(x)$ and $\mathcal{F}_L g_2(\mathcal{M}_x) \underset{\sigma}{\to} g_2(x)$, and also we have $\mathcal{F}_L g_1(\mathcal{M}_x) = \mathcal{F}_L g_2(\mathcal{M}_x) = \mathcal{F}_L f(\mathcal{M}_x)$ an L-filter on Y, and since (Y, σ) is a GT_2-space, then $g_1(x) = g_2(x)$. Thus $g_1 = g_2$. $\square$

Lemma 4.4 *An extension* (Y, σ) *of* (X, τ) *is reduced* if and only if $\mathcal{M}_x \neq \mathcal{M}_y$ *for all* $x \neq y$ *in* Y *and* $x \in Y \backslash X$*.*

Proof. The proof comes from that

$$\mathcal{M}_x = \mathcal{N}_\sigma(x) \wedge \dot{X} \neq \mathcal{N}_\sigma(y) \wedge \dot{X} = \mathcal{M}_y$$

if and only if $\mathcal{N}_\sigma(x) \neq \mathcal{N}_\sigma(y)$. $\square$

Definition 4.3 An L-uniform space $(Y, \mathcal{U}^*)$ is called an *extension* of the L-uniform space $(X, \mathcal{U})$ if $X \subseteq Y$, $\mathcal{U} = \mathcal{U}^*_X$ and X is a $\tau_{\mathcal{U}^*}$-dense in Y.

Definition 4.4 An L-uniform space $(Y, \mathcal{U}^*)$ is called a *reduced extension* of the L-uniform space $(X, \mathcal{U})$ if $(Y, \tau_{\mathcal{U}^*})$ is a reduced extension of $(X, \tau_{\mathcal{U}})$.

An L-uniform structure $\mathcal{U}$ on a set X is called *separated* [5] if for all $x, y \in X$ with $x \neq y$ there is $u \in L^{X \times X}$ such that $\mathcal{U}(u) = 1$ and $u(x, y) = 0$. The space $(X,\mathcal{U})$ is called *separated L-uniform space.*

Proposition 4.2 [5] *Let X be a set, $\mathcal{U}$ an L-uniform structure on X and $\tau_{\mathcal{U}}$ the L-topology associated with $\mathcal{U}$. Then*

$$(X,\mathcal{U}) \textit{ is separated} \text{ if and only if } (X, \tau_{\mathcal{U}}) \textit{ is } GT_0\textit{-space.}$$

Lemma 4.5 *If $(X,\mathcal{U})$ is a separated L-uniform space and $(Y,\mathcal{U}^*)$ is a reduced extension of $(X,\mathcal{U})$, then $(Y,\mathcal{U}^*)$ is separated as well.*

Proof. From Proposition 4.2, we get that $(X, \tau_{\mathcal{U}})$ is a GT_0-space and since $(Y, \tau_{\mathcal{U}^*})$ is a reduced extension of $(X, \tau_{\mathcal{U}})$, then by Lemma 4.2 we have $(Y, \tau_{\mathcal{U}^*})$ is a GT_0-space. Again by Proposition 4.2, we get that $(Y,\mathcal{U}^*)$ is separated. □

Now, we give this definition.

Definition 4.5 An L-uniform space $(X,\mathcal{U})$ is called *complete* if every $\mathcal{U}$-cauchy filter $\mathcal{M}$ on X is convergent.

Definition 4.6 An L-uniform space $(Y,\mathcal{U}^*)$ is called a *completion* of the L-uniform space $(X,\mathcal{U})$ if it is a reduced extension of $(X,\mathcal{U})$ and $\mathcal{U}^*$ is complete.

Lemma 4.6 *The completion of a separated L-uniform space is separated as well.*

Proof. The proof comes from Lemma 4.5. □

5. The completion of L-topological groups

In this section, we introduce the main notion of this paper, that the completion of L-topological groups using the completion of L-uniform spaces.

L-topological groups. Let G be a multiplicative group. We denote, as usual, the identity element of G by e and the inverse of an element a of G by a^{-1}.

Definition 5.1 [1, 6] Let G be a group and τ an L-topology on G. Then (G, τ) will be called an *L-topological group* if the mappings

$$\pi : (G \times G, \tau \times \tau) \to (G, \tau) \;\text{ defined by }\; \pi(a, b) = ab \;\text{ for all }\; a, b \in G$$

and

$$i : (G, \tau) \to (G, \tau) \;\text{ defined by }\; i(a) = a^{-1} \;\text{ for all }\; a \in G$$

are L-continuous. π and i are the binary operation and the unary operation of the inverse on G, respectively.

For all $\lambda \in L^G$, denote by λ^i the L-set $\lambda \circ i$ in G, that is, $\lambda^i(x) = \lambda(x^{-1})$ for all $x \in G$. We also denote $\mathcal{F}_L\pi(\mathcal{L} \times \mathcal{M})$ by $\mathcal{LM}$ and $\mathcal{F}_L i(\mathcal{M})$ by $\mathcal{M}^i$, which means that $\mathcal{LM}(\lambda) = \mathcal{L} \times \mathcal{M}(\lambda \circ \pi)$ and $\mathcal{M}^i(\lambda) = \mathcal{M}(\lambda^i)$ for all L-filters $\mathcal{L}, \mathcal{M}$ on G and all L-sets $\lambda \in L^G$.

A surrounding $u \in L^{X \times X}$ is called *left* (*right*) *invariant* provided

$$u(ax, ay) = u(x, y) \qquad (u(xa, ya) = u(x, y)) \text{ for all } a, x, y \in X.$$

$\mathcal{U}$ is called a *left* (*right*) *invariant* L-uniform structure if $\mathcal{U}$ has a valued L-filter base consists of left (right) invariant surroundings [8].

Proposition 5.1 [8] *Let (G, τ) be an L-topological group. Then there exist on G a unique left invariant L-uniform structure $\mathcal{U}^l$ and a unique right invariant L-uniform structure $\mathcal{U}^r$ compatible with τ, constructed using the family $(\alpha\text{-pr}\,\mathcal{N}(e))_{\alpha \in L_0}$ of all filters α-pr$\,\mathcal{N}(e)$, where $\mathcal{N}(e)$ is the L-neighborhood filter at the identity element e of (G, τ), as follows:*

$$\mathcal{U}^l(u) = \bigvee_{v \in \mathcal{U}^l_\alpha,\, v \leq u} \alpha \qquad \text{and} \qquad \mathcal{U}^r(u) = \bigvee_{v \in \mathcal{U}^r_\alpha,\, v \leq u} \alpha, \tag{5.1}$$

where

$$\mathcal{U}^l_\alpha = \alpha\text{-pr}\,\mathcal{U}^l \qquad \text{and} \qquad \mathcal{U}^r_\alpha = \alpha\text{-pr}\,\mathcal{U}^r \tag{5.2}$$

are defined by

$$\mathcal{U}^l_\alpha = \{u \in L^{G \times G} \mid u(x, y) = (\lambda \wedge \lambda^i)(x^{-1}y) \text{ for some } \lambda \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{5.3}$$

and

$$\mathcal{U}^r_\alpha = \{u \in L^{G \times G} \mid u(x, y) = (\lambda \wedge \lambda^i)(xy^{-1}) \text{ for some } \lambda \in \alpha\text{-pr}\,\mathcal{N}(e)\} \tag{5.4}$$

We should notice that we shall fix the notations $\mathcal{U}^l$, $\mathcal{U}^r$, $\mathcal{U}^l_\alpha$ and $\mathcal{U}^r_\alpha$ along the paper to be these defined above.

Definition 5.2 $\mathcal{U}^b = \mathcal{U}^l \vee \mathcal{U}^r$ is called the *bilateral* L-uniform structure of the L-topological group (G, τ), where $\mathcal{U}^l$ and $\mathcal{U}^r$ are defined in (5.1) - (5.4).

Remark 5.1 $\mathcal{M}$ is a $\mathcal{U}^b$-cauchy filter if it is $\mathcal{U}^l$-cauchy filter and $\mathcal{U}^r$-cauchy filter simultaneously.

Remark 5.2 (cf. [8]) For the L-topological group (G, τ), the elements of $\mathcal{U}^l_\alpha$ ($\mathcal{U}^r_\alpha$) are left (right) invariant surroundings. Moreover, $(\mathcal{U}^l_\alpha)_{\alpha \in L_0}$ ($(\mathcal{U}^r_\alpha)_{\alpha \in L_0}$) is a valued L-filter base for the left (right) invariant L-uniform structure $\mathcal{U}^l$ ($\mathcal{U}^r$) defined by (5.1) - (5.4), respectively.

Now, suppose that (G, τ) has a countable L-neighborhood filter $\mathcal{N}(e)$ at the identity e. Since any L-topological group, from Proposition 5.1, is uniformizable, then the left and the right invariant L-uniform structures $\mathcal{U}^l$ and $\mathcal{U}^r$, constructed also in Proposition 5.1, has, from Remark 5.2, a countable L-filter base $\mathcal{U}^l_{\frac{1}{n}}$ and $\mathcal{U}^r_{\frac{1}{n}}$, respectively, $n \in \mathbf{N}$.

We may recall that if (G, τ) is an L-topological group and A is a subgroup of G, then the L-topological subspace (A, τ_A) is called an *L-topological subgroup* [6].

Proposition 5.2 *Let (A, τ_A) be an L-topological subgroup of an L-topological group (G, τ), and further $\mathcal{U}$ be a complete L-uniform structure on G compatible with τ and $\mathcal{U}_A$ is the L-uniform structure on A compatible with τ_A. Then*

(d1) *If $\mathcal{L}$ and $\mathcal{M}$ are $\mathcal{U}_A$-cauchy filters, then $\mathcal{LM}$ is a $\mathcal{U}_A$-cauchy filter as well,*

(d2) *If $\mathcal{M}$ is a $\mathcal{U}_A$-cauchy filter, then $\mathcal{M}^i$ is a $\mathcal{U}_A$-cauchy filter as well.*

Proof. By Lemma 3.1, $\mathcal{L}$ and $\mathcal{M}$ are both $\mathcal{U}$-cauchy filters too, thus $\mathcal{U}$ is complete implies $\mathcal{L} \underset{\tau}{\to} x$ and $\mathcal{M} \underset{\tau}{\to} y$ for some $x, y \in G$, that is, $\mathcal{L} \leq \mathcal{N}(x)$ and $\mathcal{M} \leq \mathcal{N}(y)$. Now, for each $\xi \in L^G$ we have

$$\begin{aligned}
\mathcal{LM}(\xi) &= \mathcal{F}_L\pi(\mathcal{L} \times \mathcal{M})(\xi) \\
&= \mathcal{L} \times \mathcal{M}(\xi \circ \pi) \\
&= \bigvee_{\lambda \times \mu \leq \xi \circ \pi} \mathcal{L}(\lambda) \wedge \mathcal{M}(\mu) \\
&\geq \bigvee_{\lambda \times \mu \leq \xi \circ \pi} \mathcal{N}(x)(\lambda) \wedge \mathcal{N}(y)(\mu) \\
&= \bigvee_{\lambda \times \mu \leq \xi \circ \pi} \mathrm{int}_\tau \lambda(x) \wedge \mathrm{int}_\tau \mu(y) \\
&\geq \mathrm{int}_\tau \xi(xy) \\
&= \mathcal{N}(xy)(\xi).
\end{aligned}$$

That is, $\mathcal{LM} \underset{\tau}{\to} xy$ and hence, $\mathcal{LM}$ is a $\mathcal{U}$-cauchy filter and at the same time a $\mathcal{U}_A$-cauchy filter from Proposition 3.1 and Lemma 3.1.

Similarly, if $\mathcal{M}$ is a $\mathcal{U}_A$-cauchy filter, and thus a $\mathcal{U}$-cauchy filter, then $\mathcal{M} \underset{\tau}{\to} x$, and hence by Lemma 2.1, $\mathcal{M}^i(\lambda) = \mathcal{F}_L i(\mathcal{M}) \underset{\tau}{\to} i(x) = x^{-1}$. This means that $\mathcal{M}^i$ is a $\mathcal{U}$-cauchy filter and also a $\mathcal{U}_A$-cauchy filter. □

Definition 5.3 Let us call an L-uniform structure $\mathcal{U}$ of an L-topological group (G, τ) *admissible* if $\tau_{\mathcal{U}} = \tau$ and the conditions (d1) and (d2) in Proposition 5.2 are fulfilled.

Definition 5.4 An L-topological group (G, τ) is called *complete* if its bilateral L-uniform structure $\mathcal{U}^b$ is complete. (G, τ) is called *left complete* (*right complete*) if it is complete and its left (right) L-uniform structure $\mathcal{U}^l$ ($\mathcal{U}^r$) is admissible.

Lemma 5.1 *The inverse mapping $i : (G, \tau) \to (G, \tau)$, $i(x) = x^{-1}$, of any L-topological group (G, τ) is $(\mathcal{U}^l,\mathcal{U}^r)$-continuous and $(\mathcal{U}^r,\mathcal{U}^l)$-continuous, and moreover $\mathcal{U}^r = \mathcal{F}_L(i \times i)(\mathcal{U}^l)$, $\mathcal{U}^l = \mathcal{F}_L(i \times i)(\mathcal{U}^r)$.*

Proof. For $u \in \mathcal{U}^l_\alpha$ and for some $\lambda \in \alpha$ - pr $\mathcal{N}(e)$, we have

$$(u \circ (i \times i))(x, y) = u(x^{-1}, y^{-1}) = (\lambda \wedge \lambda^i)(xy^{-1}) = w(x, y)$$

for some $w \in \mathcal{U}^r_\alpha$. Since $\mathcal{F}_L(i \times i)(\mathcal{U}^l)(u) = \mathcal{U}^l(u \circ (i \times i))$ for all $u \in L^{X\times X}$, then $\mathcal{F}_L(i\times i)(\mathcal{U}^l)(u) = \mathcal{U}^r(u)$ for all $u \in L^{X\times X}$, and hence i is a $(\mathcal{U}^l,\mathcal{U}^r)$-continuous. Similarly, we get that $\mathcal{F}_L(i \times i)(\mathcal{U}^r) = \mathcal{U}^l$, and thus i is a $(\mathcal{U}^r,\mathcal{U}^l)$-continuous. □

Proposition 5.3 *If $\mathcal{M}$ is a $\mathcal{U}^l$-cauchy filter in an L-topological group (G,τ), then $\mathcal{M}^i$ is a $\mathcal{U}^r$-cauchy filter and the converse.*

Proof. Since, from Lemma 5.1, the mapping $i : (G,\mathcal{U}^l) \to (G,\mathcal{U}^r)$ is $(\mathcal{U}^l,\mathcal{U}^r)$-continuous, then $\mathcal{M}$ is a $\mathcal{U}^l$-cauchy filter implies, from Lemma 3.2, that $\mathcal{F}_L(i)(\mathcal{M}) = \mathcal{M}^i$ is a $\mathcal{U}^r$-cauchy filter. Similarly, the converse follows. □

Proposition 5.4 [15] *Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two L-uniform spaces and $f : X \to Y$ a mapping. Then the mapping $f : (X,\tau_{\mathcal{U}}) \to (Y,\tau_{\mathcal{V}})$ is L-continuous* if and only if *f is $(\mathcal{U},\mathcal{V})$-continuous.*

Here, we give this result.

Lemma 5.2 *If $\mathcal{U}$ and $\mathcal{V}$ are two L-uniform structures on an L-topological group (G,τ) and both $\mathcal{L}$ and $\mathcal{M}$ are $\mathcal{U}$- ($\mathcal{V}$-)cauchy filters on G, then $\mathcal{L}\times\mathcal{M}$ is a $\mathcal{U}\times\mathcal{U}$- ($\mathcal{V}\times\mathcal{V}$-)cauchy filter on $G \times G$.*

Proof. From Proposition 2.2, $\mathcal{L} \times \mathcal{M}$ is an L-filter on $G \times G$. Let $\mathcal{L}$ and $\mathcal{M}$ be $\mathcal{U}$-cauchy filters on G, then there exist $A, B \subseteq G$ such that $\mathcal{L} \le \dot{A}$ and $\mathcal{M} \le \dot{B}$ and A, B are small of order every surrounding u in $(G,\mathcal{U})$. Now,

$$\begin{aligned}(\mathcal{L}\times\mathcal{M})(u) &= \bigvee_{\lambda\times\mu\le u} (\mathcal{L}(\lambda)\wedge\mathcal{M}(\mu))\\ &\ge \bigvee_{\lambda\times\mu\le u} (\dot{A}(\lambda)\wedge\dot{B}(\mu))\\ &= \bigvee_{\lambda\times\mu\le u}\ \bigwedge_{x\in A,\, y\in B} \lambda(x)\wedge\mu(y)\\ &= \bigvee_{\lambda\times\mu\le u}\ \bigwedge_{x\in A,\, y\in B} \lambda\times\mu(x,y)\\ &= u(A,B)\\ &= (A \dot{\times} B)(u)\end{aligned}$$

for all $u \in L^{G\times G}$. That is, there exists $A \times B \subseteq G \times G$ such that $\mathcal{L} \times \mathcal{M} \le (A \dot{\times} B)$.

Let $\psi : (G \times G) \times (G \times G) \to L$ be a mapping and u a surrounding in $(G,\mathcal{U})$, then from Proposition 5.4, π is $(\mathcal{U} \times \mathcal{U},\mathcal{U})$-continuous, and then

$$\alpha \le \mathcal{U}(u) \le \mathcal{F}_L(\pi\times\pi)(\mathcal{U}\times\mathcal{U})(u) = \mathcal{U}\times\mathcal{U}(u\circ(\pi\times\pi)) = \mathcal{U}\times\mathcal{U}(\psi)$$

and also, $u = u^{-1}$ implies that

$$\psi^{-1} = (u\circ(\pi\times\pi))^{-1} = u^{-1}\circ(\pi\times\pi) = u\circ(\pi\times\pi) = \psi,$$

that is, ψ is a surrounding in $(G\times G,\mathcal{U}\times\mathcal{U})$, and for any surrounding ψ in $(G\times G,\mathcal{U}\times\mathcal{U})$, there exists a surrounding u in $(G,\mathcal{U})$ such that $\psi = u\circ(\pi\times\pi)$.

Now, $\alpha \leq u(x,y)$ for all $x,y \in A$ and $\beta \leq u(r,s)$ for all $r,s \in B$ and for some $\alpha,\beta \in L_0$ imply that $\psi((x,r),(y,s)) = (u\circ(\pi\times\pi))((x,r),(y,s)) = u(xr,ys)$, and by choosing $(x,y)=(e,e)$ or $(r,s)=(e,e)$, we get that $u(xr,ys)\geq\gamma$ for some $\gamma\in L_0$, that is, for all $(x,r),(y,s)\in A\times B$, we have $\psi((x,r),(y,s))\geq\gamma$ for some $\gamma\in L_0$, which means that $A\times B$ is small of order every surrounding in $(G\times G,\mathcal{U}\times\mathcal{U})$, and therefore $\mathcal{L}\times\mathcal{M}$ is a $\mathcal{U}\times\mathcal{U}$-cauchy filter. $\square$

Proposition 5.5 *If $\mathcal{U}^l$ and $\mathcal{U}^r$ are the left and the right L-uniform structures of an L-topological group (G,τ) and both of $\mathcal{L}$ and $\mathcal{M}$ are $\mathcal{U}^l$- ($\mathcal{U}^r$-)cauchy filters, then $\mathcal{L}\mathcal{M}$ has the same property.*

Proof. From Lemma 5.2 and Lemma 3.2, we have $\mathcal{L}\mathcal{M}=\mathcal{F}_L\pi(\mathcal{L}\times\mathcal{M})$ is a $\mathcal{U}^l$- ($\mathcal{U}^r$-)cauchy filter. $\square$

Accordingly, the property of being admissible depends for $\mathcal{U}^l$ and $\mathcal{U}^r$ on the fact whether condition (d2) of Proposition 5.2 is fulfilled.

Proposition 5.6 *The following statements are equivalent in any L-topological group (G,τ).*

(1) *Together with $\mathcal{M}$, $\mathcal{M}^i$ is a $\mathcal{U}^l$-cauchy filter,*

(2) *Together with $\mathcal{M}$, $\mathcal{M}^i$ is a $\mathcal{U}^r$-cauchy filter,*

(3) *Every $\mathcal{U}^l$-cauchy filter is a $\mathcal{U}^r$-cauchy filter,*

(4) *Every $\mathcal{U}^r$-cauchy filter is a $\mathcal{U}^l$-cauchy filter,*

(5) *$\mathcal{U}^l$ is admissible,*

(6) *$\mathcal{U}^r$ is admissible.*

Proof. (1) $\Longleftrightarrow$ (5) and (2) $\Longleftrightarrow$ (6) come from Proposition 5.5.

(1) $\Longleftrightarrow$ (2) follows from Proposition 5.3 and that $(\mathcal{M}^i)^i=\mathcal{M}$.

From (1), since $\mathcal{M}$ is a $\mathcal{U}^l$-cauchy filter implies that $\mathcal{M}^i$ is a $\mathcal{U}^l$-cauchy filter, and thus $\mathcal{M}$ is a $\mathcal{U}^r$-cauchy filter according to Proposition 5.3, then (1) $\Longrightarrow$ (3); On the other hand, if $\mathcal{M}$ is a $\mathcal{U}^l$-cauchy filter, then it is a $\mathcal{U}^r$-cauchy filter and thus $\mathcal{M}^i$ is a $\mathcal{U}^l$-cauchy filter. That is, (1) $\Longleftrightarrow$ (3).

(2) $\Longleftrightarrow$ (4) is obtained similarly. $\square$

Proposition 5.7 *If the left L-uniform structure $\mathcal{U}^l$ or the right L-uniform structure $\mathcal{U}^r$ of an L-topological group (G,τ) is complete, then the other one is complete as well and both are admissible.*

Proof. If $\mathcal{U}^l$ is complete and $\mathcal{M}$ is a $\mathcal{U}^r$-cauchy filter, then from Proposition 5.3, $\mathcal{M}^i$ is a $\mathcal{U}^l$-cauchy filter, thus $\mathcal{M}^i \underset{\tau}{\to} x$ in G and then $\mathcal{M} \underset{\tau}{\to} x^{-1}$. Hence, $\mathcal{U}^r$ is complete, and the completeness of $\mathcal{U}^l$ follows by the same way from the completeness of $\mathcal{U}^r$.

At last, $\mathcal{M}$ is a $\mathcal{U}^l$-cauchy filter implies that $\mathcal{M}$ converges to $x \in G$, that is, $\mathcal{M} \le \mathcal{U}^l[\dot{x}]$, and then $\mathcal{M}^i \le \mathcal{U}^l[\dot{x^{-1}}]$ and from Proposition 3.1, $\mathcal{M}^i$ is a $\mathcal{U}^l$-cauchy filter. Proposition 5.6 implies that both $\mathcal{U}^l$ and $\mathcal{U}^r$ are admissible. □

Lemma 5.3 *If $\mathcal{U}^b$ is the bilateral L-uniform structure of an L-topological group (G, τ), then i is $(\mathcal{U}^b,\mathcal{U}^b)$-continuous.*

Proof. From that $\mathcal{U}^l \le \mathcal{U}^b$ and $\mathcal{U}^r \le \mathcal{U}^b$, we get that $\mathcal{F}_L(i \times i)\mathcal{U}^l \le \mathcal{U}^b$ and $\mathcal{F}_L(i \times i)\mathcal{U}^r \le \mathcal{U}^b$, and thus

$$\mathcal{F}_L(i \times i)\mathcal{U}^b = \mathcal{F}_L(i \times i)\mathcal{U}^l \vee \mathcal{F}_L(i \times i)\mathcal{U}^r \le \mathcal{U}^b.$$

Hence, i is $(\mathcal{U}^b,\mathcal{U}^b)$-continuous. □

***L*-metric spaces.** We use here the notion of L-metric space defined by means of the notion of L-real numbers in [12]. By an L-real number is meant [12] a convex, normal, compactly supported and upper semi-continuous L-subset of the set of all real numbers $\mathbf{R}$. The set of all L-real numbers is denoted by $\mathbf{R}_L$. $\mathbf{R}$ is canonically embedded into $\mathbf{R}_L$, identifying each real number a with the crisp L-number $a^{\sim}$ defined by $a^{\sim}(\xi) = 1$ if $\xi = a$ and 0 otherwise. The set of all positive L-real numbers is defined and denoted by: $\mathbf{R}^*_L = \{x \in \mathbf{R}_L \mid x(0) = 1 \text{ and } 0^{\sim} \le x\}$ [12].

A mapping $\varrho : X \times X \longrightarrow \mathbf{R}^*_L$ is called an *L-metric* [12] on X if the following conditions are fulfilled:

(1) $\varrho(x, y) = 0^{\sim}$ if and only if $x = y$

(2) $\varrho(x, y) = \varrho(y, x)$

(3) $\varrho(x, y) \le \varrho(x, z) + \varrho(z, y)$.

If $\varrho : X \times X \longrightarrow \mathbf{R}^*_L$ satisfied the conditions (2) and (3) and the following condition:

$(1)'$ $\varrho(x, y) = 0^{\sim}$ if $x = y$

then it is called an *L-pseudo-metric* on X.

A set X equipped with an L-pseudo-metric (L-metric) ϱ on X is called an *L-pseudo-metric (L-metric) space*.

To each L-pseudo-metric (L-metric) ϱ on a set X is generated canonically a stratified L-topology τ_ϱ on X which has $\{\varepsilon \circ \varrho_x \mid \varepsilon \in \mathcal{E},\ x \in X\}$ as a base, where $\varrho_x : X \to \mathbf{R}^*_L$ is the mapping defined by $\varrho_x(y) = \varrho(x, y)$ and

$$\mathcal{E} \;=\; \{\,\overline{\alpha} \wedge R^{\delta}|_{\mathbf{R}^*_L} \;\mid\; \delta > 0,\ \alpha \in L\,\} \cup \{\,\overline{\alpha} \;\mid\; \alpha \in L\,\},$$

here $\overline{\alpha}$ has $\mathbf{R}^*_L$ as domain.

An L-topological space (X, τ) is called *pseudo-metrizable* (*metrizable*) if there is an L-pseudo-metric (L-metric) ϱ on X inducing τ, that is, $\tau = \tau_\varrho$.

An L-pseudo-metric ϱ is called *left* (*right*) *invariant* if

$$\varrho(x,y) = \varrho(ax,ay) \qquad (\varrho(x,y) = \varrho(xa,ya)) \text{ for all } a,x,y \in X.$$

An L-topological group (G, τ) is called *separated* if for the identity element e, we have $\bigwedge_{\lambda \in \alpha\text{-pr}\mathcal{N}(e)} \lambda(e) \geq \alpha$, and $\bigwedge_{\lambda \in \alpha\text{-pr}\mathcal{N}(e)} \lambda(x) < \alpha$ for all $x \in G$ with $x \neq e$ and for all $\alpha \in L_0$ [8].

Proposition 5.8 [9] *Let (G, τ) be a* (separated) *L-topological group. Then the following statements are equivalent.*

(1) *τ is pseudo-metrizable* (metrizable)*;*

(2) *e has a countable L-neighborhood filter $\mathcal{N}(e)$;*

(3) *τ can be induced by a left invariant L-pseudo-metric* (L-metric)*;*

(4) *τ can be induced by a right invariant L-pseudo-metric* (L-metric)*.*

Definition 5.5 An L-uniform structure $\mathcal{U}$ on a set X is called *pseudo-metrizable* (*metrizable*) if there exists a countable L-uniform base for $\mathcal{U}$ (and $\mathcal{U}$ is separated).

Proposition 5.9 [8] *Let (G, τ) be an L-topological group. Then there exist on G a unique left invariant L-uniform structure $\mathcal{U}^l$ and a unique right invariant L-uniform structure $\mathcal{U}^r$ compatible with τ, constructed with* (5.1) - (5.4)*.*

Proposition 5.10 *For any* (separated) *L-topological group (G, τ), The L-uniform structures $\mathcal{U}^l$, $\mathcal{U}^r$ and $\mathcal{U}^b$ constructed in* (5.1) - (5.4) *are pseudo-metrizable* (metrizable)*.*

Proof. From Proposition 5.8, $\tau = \tau_{\varrho_1} = \tau_{\varrho_2}$ where ϱ_1 is a left, ϱ_2 is a right invariant L-pseudo-metric (L-metric) on G, and then $\mathcal{U}_{\varrho_1}$ is left invariant and $\mathcal{U}_{\varrho_2}$ is right invariant. From Proposition 5.9, $\mathcal{U}^l$ and $\mathcal{U}^r$ are unique, that is, $\mathcal{U}_{\varrho_1} = \mathcal{U}^l$, $\mathcal{U}_{\varrho_2} = \mathcal{U}^r$ and $\mathcal{U}^l$, $\mathcal{U}^r$ are pseudo-metrizable (metrizable). Moreover, $\tau_{\mathcal{U}^b} = \tau_{\mathcal{U}^l \vee \mathcal{U}^r} = \tau_{\mathcal{U}^l} \vee \tau_{\mathcal{U}^r} = \tau$. Hence, $\mathcal{U}^b$ is pseudo-metrizable (metrizable) as well. □

Proposition 5.11 [4] *Let $(X, \mathcal{U})$ be an L-uniform space, $(A, \mathcal{U}_A)$ an L-uniform subspace of $(X, \mathcal{U})$ and $(\tau_\mathcal{U})_A$ the L-subtopology of the L-topology $\tau_\mathcal{U}$ associated with $\mathcal{U}$. Then the L-topology associated to $\mathcal{U}_A$ coincides with $(\tau_\mathcal{U})_A$, that is, $\tau_{(\mathcal{U}_A)} = (\tau_\mathcal{U})_A$.*

Lemma 5.4 *Let (A, τ_A) be an L-topological subgroup of an L-topological subgroup (G, τ), and $\mathcal{U}^l$, $\mathcal{U}^r$ and $\mathcal{U}^b$ the left, the right and the bilateral L-uniform structures of (G, τ). Then the corresponding L-uniform structures of (A, τ_A) are $(\mathcal{U}^l)_A$, $(\mathcal{U}^r)_A$ and $(\mathcal{U}^b)_A$, respectively.*

Proof. From Proposition 5.11, we have $\tau_{(\mathcal{U}^l)_A} = (\tau_{\mathcal{U}^l})_A = \tau_A$ and, together with $\mathcal{U}^l$, $(\mathcal{U}^l)_A$ is left invariant as well, and hence $(\mathcal{U}^l)_A$ is the left invariant L-uniform structure of (A, τ_A). By the same $(\mathcal{U}^r)_A$ is the right invariant L-uniform structure of (A, τ_A) as well. Moreover,

$$\tau_{\mathcal{U}^b_A} = \tau_{(\mathcal{U}^l_A \vee \mathcal{U}^r_A)} = \tau_{\mathcal{U}^l_A} \vee \tau_{\mathcal{U}^r_A} = (\tau_{\mathcal{U}^l})_A \vee (\tau_{\mathcal{U}^r})_A = (\tau_{\mathcal{U}^b})_A = \tau_A. \ \Box$$

Here, we give the essential result in this section.

Definition 5.6 For a separated L-topological group (G, τ), let us call (H, σ) a *completion* of (G, τ) if it is complete separated L-topological group and in which (G, τ) is a dense subgroup.

In the following we need this result.

Proposition 5.12 [8] *Let (G, τ) be an L-topological group. Then the following statements are equivalent.*

(1) *The L-topology τ is GT_0.*

(2) *The L-topology τ is GT_2.*

(3) *The L-topological group (G, τ) is separated.*

Proposition 5.13 *Let (G, τ) be a separated L-topological group, $\mathcal{U}$ an admissible L-uniform structure on G, and $(H, \mathcal{V})$ the completion of $(G, \mathcal{U})$. Then an operation $\pi' : H \times H \to H$ can be defined on H in a unique way so that H equipped with π' is a group, and $(H, \tau_{\mathcal{V}})$ is an L-topological group of which G is a subgroup.*

Proof. Let $\sigma = \tau_{\mathcal{V}}$. If $\pi' : H \times H \to H$ is defined by $\pi'(y, z) = yz$ for all $y, z \in H$, then $\pi'|_{G\times G} = \pi$. Now, let $\mathcal{L}_x$ and $\mathcal{M}_y$ be two trace filters on H at x and y into H, respectively. Since $\mathcal{L}_x \underset{\sigma}{\to} x$ and $\mathcal{M}_y \underset{\sigma}{\to} y$, that is, $\mathcal{L}_x(\lambda) \geq \mathrm{int}_\sigma \lambda(x)$ and $\mathcal{M}_y(\mu) \geq \mathrm{int}_\sigma \mu(y)$, then

$$\begin{aligned}
\mathcal{L}_x\mathcal{M}_y(\xi) &= \mathcal{F}_L\pi'(\mathcal{L}_x \times \mathcal{M}_y)(\xi) \\
&= \mathcal{L}_x \times \mathcal{M}_y(\xi \circ \pi') \\
&= \bigvee_{\lambda\times\mu\leq\xi\circ\pi'} \mathcal{L}_x(\lambda) \wedge \mathcal{M}_y(\mu) \\
&\geq \bigvee_{\lambda\times\mu\leq\xi\circ\pi'} \mathrm{int}_\sigma\lambda(x) \wedge \mathrm{int}_\sigma\mu(y) \\
&\geq \mathrm{int}_\sigma\xi(xy) \\
&= \mathcal{N}_\sigma(xy)(\xi),
\end{aligned}$$

and then $\mathcal{L}_x\mathcal{M}_y \underset{\sigma}{\to} xy$. From that $\mathcal{U}$ is separated and from Lemma 4.6 and Proposition 5.12, we get (H, σ) is a GT_2-space, and therefore these properties, using Lemma 4.3

and Remark 4.2, define π' in a unique way as the only continuous extension of π into $H \times H$. Also, if $i' : H \to H$ is defined by $i'(y) = y^{-1}$ for all $y \in H$, then $i'|_G = i$ and $\mathcal{F}_L i'(\mathcal{L}_x) = \mathcal{L}_x^{i'} \underset{\sigma}{\to} x^{-1}$ for any trace filter $\mathcal{L}_x$ on H, and i' is (σ, σ)-continuous, that is, as in before, i' is a continuous extension of i defined in a unique manner.

Hence, π' is $(\sigma \times \sigma, \sigma)$-continuous and i' is (σ, σ)-continuous imply that (H, σ) is an L-topological group in which (G, τ) is an L-topological subgroup. □

Proposition 5.14 *Under the hypothesis of* Proposition 5.13*, if the left, the right or the bilateral L-uniform structure of $(H, \tau_{\mathcal{U}^*})$ is $\mathcal{U}^{*l}$, $\mathcal{U}^{*r}$, or $\mathcal{U}^{*b}$ respectively, then the corresponding L-uniform structures of (G, τ) are $(\mathcal{U}^{*l})_G$, $(\mathcal{U}^{*r})_G$, or $(\mathcal{U}^{*b})_G$.*

Proof. It is a consequence of Lemma 5.4. □

Proposition 5.15 *Let (G, τ) be a separated L-topological group, $\mathcal{U}^b$ its bilateral L-uniform structure, and $(H, \sigma = \tau_{\mathcal{V}})$ the L-topological group constructed in* Proposition 5.13 *with the choice $\mathcal{V} = \mathcal{V}^b$. Then (H, σ) is a completion of (G, τ).*

Proof. If $\mathcal{U} = \mathcal{U}^b$, then Proposition 5.13 can be applied and $\mathcal{U}^b$ is admissible where both of $\mathcal{U}^l$ and $\mathcal{U}^r$ are admissible. Also, $\mathcal{V}$ is a complete separated L-uniform structure such that $\sigma = \tau_{\mathcal{V}}$, G is σ-dense in H and $(\mathcal{V})_G = \mathcal{U}^b$. On the other hand, by Proposition 5.14, for the bilateral L-uniform structure $\mathcal{V}^b$ of the L-topological group (H, σ) we have $\sigma = \tau_{(\mathcal{V}^b)}$ and $(\mathcal{V}^b)_G = \mathcal{U}^b$. Therefore, the bilateral L-uniform structure $\mathcal{V}^b$ of (H, σ) is complete and (H, σ) is a completion of (G, τ). □

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